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Technical Notes

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Low-Diffusion Efficient Upwind Scheme

Ge-Cheng Zha*
University of Miami, Coral Gables, Florida 33124

I. Introduction

DEVELOPMENT of an accurate and efficient numerical scheme for compressible flow governing equations is essential because of the increasing engineering demand for aircraft and spacecraft design.[1] An accurate, efficient, and robust upwind scheme used as the Riemann solver to resolve shock waves and wall boundary layers is very important.

Zha and Hu recently suggested an E-CUSP scheme, which has low diffusion and can capture crisp shock-wave profiles and exact contact discontinuities.[2] The scheme is consistent with the characteristic directions caused by the nature of E-CUSP scheme. The scheme shows the highest stability for two shock-tube test problems compared with several other popularly used upwind schemes for the explicit Euler time-marching scheme. The scheme also works well when extended to multiple dimensions.[2] However, with more and more applications, it is found that the E-CUSP scheme of Zha–Hu can generate temperature oscillation near computational boundaries.

This Note suggests a modification by replacing the pressure term with total enthalpy in the dissipation term of the energy equation. The temperature oscillation is removed, and low numerical dissipation is achieved. The modified scheme yields smooth temperature field near boundaries and more precise wall temperature than the original scheme. The scheme is shown to be accurate, robust, and efficient by the cases tested in this Note and other three-dimensional applications.[3,4]

For the terminology simplicity and following the terms used in the scheme development, the original E-CUSP scheme[2] is named Zha CUSP scheme, and the modified scheme is named as Zha CUSP2 scheme.

II. Numerical Scheme

A. Governing Equations

To describe the new scheme, we begin with the quasi-one-dimensional Euler equations in Cartesian coordinates for inviscid flow:

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{E} - \boldsymbol{H} = 0 \tag{1}$$

where

$$\boldsymbol{U} = S\boldsymbol{Q}, \qquad \boldsymbol{Q} = \begin{pmatrix} \rho \\ \rho u \\ \rho e \end{pmatrix}, \qquad \boldsymbol{E} = S\boldsymbol{F}$$

$$\boldsymbol{F} = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ (\rho e + p)u \end{pmatrix}, \qquad \boldsymbol{H} = \frac{\mathrm{d}S}{\mathrm{d}x}\begin{pmatrix} 0 \\ p \\ 0 \end{pmatrix} \tag{2}$$

In the preceding equations, ρ is the density, u is the velocity, p is the static pressure, e is the total energy per unit mass, and S is the cross-sectional area of the one-dimensional duct. The following state equation is also employed:

$$p = (\gamma - 1)\left(\rho e - \tfrac{1}{2}\rho u^2\right) \tag{3}$$

where γ is the specific heat ratio with the value of 1.4 for ideal gas.

The finite volume method with the explicit Euler temporal integration is used to discretize the governing equations. A numerical scheme is needed to evaluate the interface flux:

$$\boldsymbol{E}_{i+\frac{1}{2}} = S\boldsymbol{F}_{i+\frac{1}{2}} \tag{4}$$

B. Original E-CUSP Scheme (Zha CUSP)[2]

In Ref. 2, the characteristic analysis is given as the foundation to construct the original E-CUSP scheme. Herein the original scheme is directly given here:

For $|u| \le a$,

$$\boldsymbol{F}_{\frac{1}{2}} = \frac{1}{2}\left[(\rho u)_{\frac{1}{2}}\left(\boldsymbol{q}^c_L + \boldsymbol{q}^c_R\right) - |\rho u|_{\frac{1}{2}}\left(\boldsymbol{q}^c_R - \boldsymbol{q}^c_L\right)\right] + \begin{pmatrix} 0 \\ \mathcal{P}^+ p \\ \frac{1}{2}p(u + a_{\frac{1}{2}}) \end{pmatrix}_L + \begin{pmatrix} 0 \\ \mathcal{P}^- p \\ \frac{1}{2}p(u - a_{\frac{1}{2}}) \end{pmatrix}_R \tag{5}$$

where the interface mass flux is evaluated as

$$(\rho u)_{\frac{1}{2}} = \left(\rho_L u_L^+ + \rho_R u_R^-\right) \tag{6}$$

$$\boldsymbol{q}^c = \begin{pmatrix} 1 \\ u \\ e \end{pmatrix} \tag{7}$$

$$u_L^+ = a_{\frac{1}{2}}\left\{\frac{M_L + |M_L|}{2} + \alpha_L\left[\frac{1}{4}(M_L + 1)^2 - \frac{M_L + |M_L|}{2}\right]\right\} \tag{8}$$

$$u_R^- = a_{\frac{1}{2}}\left\{\frac{M_R - |M_R|}{2} + \alpha_R\left[-\frac{1}{4}(M_R - 1)^2 - \frac{M_R - |M_R|}{2}\right]\right\} \tag{9}$$

α_L and α_R are evaluated as

$$\alpha_L = \frac{2(p/\rho)_L}{(p/\rho)_L + (p/\rho)_R}, \qquad \alpha_R = \frac{2(p/\rho)_R}{(p/\rho)_L + (p/\rho)_R} \tag{10}$$

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*Associate Professor, Department of Mechanical Engineering; zha@apollo.eng.miami.edu. Member AIAA.

The interface speed of sound $a_{1/2}$, Mach number, u_L^+, and u_R^+ are evaluated as

$$a_{\frac{1}{2}} = \tfrac{1}{2}(a_L + a_R) \tag{11}$$

$$M_L = u_L/a_{\frac{1}{2}}, \qquad M_R = u_R/a_{\frac{1}{2}} \tag{12}$$

The pressure splitting coefficient is

$$\mathcal{P}^{\pm} = \tfrac{1}{4}(M \pm 1)^2(2 \mp M) \pm \alpha M(M^2 - 1)^2, \qquad \alpha = 3/16 \tag{13}$$

For $u > a$, $\boldsymbol{F}_{1/2} = \boldsymbol{F}_L$; for $u < -a$, $\boldsymbol{F}_{1/2} = \boldsymbol{F}_R$.

C. Modified Scheme (Zha CUSP2)

To cure the temperature oscillation near boundaries, the coefficient α is modified for the energy equation by replacing the pressure p with the total enthalpy h_t:

$$\alpha_L = \frac{2(h_t/\rho)_L}{(h_t/\rho)_L + (h_t/\rho)_R}, \qquad \alpha_R = \frac{2(h_t/\rho)_R}{(h_t/\rho)_L + (h_t/\rho)_R} \tag{14}$$

The total enthalpy is evaluated as

$$h_t = e + p/\rho \tag{15}$$

The coefficient α controls the numerical dissipation of the scheme. It should be emphasized that the modification of α in Eq. (14) is only applied to the energy equation. For the mass and momentum equation, the α should still use the original formulations given in Eq. (10).

This modification removes the temperature oscillations. The numerical diffusion of this modified scheme is low at stagnation as analyzed in Ref. 2 and will be also shown by the wall boundary-layer results. Extension of the modified scheme to multidimensions is straightforward as described in Ref. 2.

The reason to modify α only for energy equation is to minimize the nonmonotonicity for a contact surface. Across a contact surface, the pressure and velocity are constant, but the total enthalpy is not constant. The original formulation of α relies on pressure, which can preserve the exact contact surface. However, the original scheme lacks of dissipation to smoothen the temperature field. Using the total enthalpy in the energy equation is essential to obtain smooth temperature. Having the total enthalpy only in α for energy equation and keeping the pressure in α for mass and momentum equations will minimize the distortion of a contact surface. The Sod shock-tube results indicate that the Zha CUSP2 scheme cannot only capture crisp shock profile, but can also resolve the contact discontinuity very well.

III. Results and Discussion

According to Godunov,[5] when there are discontinuities in the solutions, monotone behavior of a solution cannot be ensured with higher than first-order scheme. Hence, it is essential to examine the accuracy and performance of a new scheme using first-order accuracy (piecewise constant). The first order is also the lowest accuracy order to test the inherent numerical dissipation of a Riemann solver. If a scheme has lower numerical dissipation at first-order accuracy, it will have higher accuracy at higher-order scheme. For the following test cases, all of the one-dimensional cases and the two-dimensional flat-plate laminar boundary layer use first-order accuracy in space. The transonic nozzle uses third-order accuracy for the inviscid fluxes with MUSCL-type differencing,[6] and no limiter is used.

A. Shock Tubes

Figure 1 is the computed velocity and temperature profiles of the Sod shock tube using the Zha CUSP2 and Roe scheme[7] with first-order accuracy and explicit Euler scheme.[8] The maximum Courant–Friedrichs–Lewy (CFL) numbers of 0.95 is used for both the schemes. Figure 1 shows that Zha CUSP2 scheme captures the shock profile using three grid points, whereas the Roe scheme uses

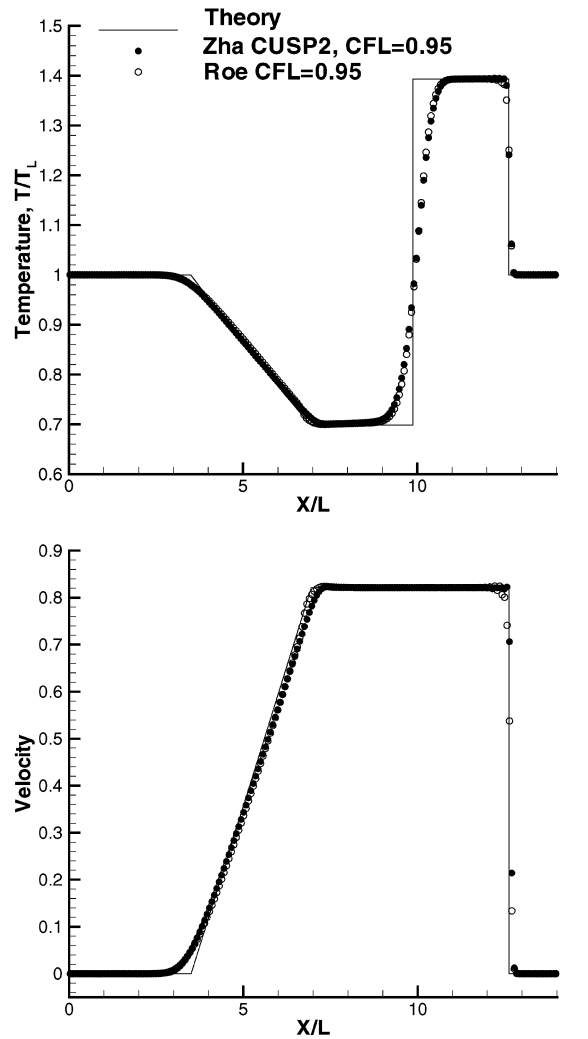


Fig. 1 Computed temperature and velocity profiles of the Sod one-dimensional shock tube using first-order schemes.

four grid points. The velocity and pressure are constant across the contact surface. The velocity profile in Fig. 1 shows that the contact discontinuity is very well resolved by both the Zha CUSP2 scheme and Roe scheme.

As pointed out in Ref. 2, the maximum CFL number limit for the Roe scheme is 0.95. Oscillations will be generated by the Roe scheme when the CFL number is greater than 0.95. The original Zha CUSP scheme can use maximum CFL number to 1.0. For the Zha CUSP2 scheme, when CFL = 1.0 is used, all of the variables are monotone except that the velocity has a small overshoot before the shock (not shown).

B. Entropy Condition

Same as the original Zha E-CUSP scheme, the modified scheme does not generate the expansion shock wave for the test case of a quasi-one-dimensional converging-diverging transonic nozzle.[2,4] The result is virtually the same as the original scheme. To save space, the results are not presented in this Note and can be seen in Ref. 4.

C. Wall Boundary Layer

To examine the numerical dissipation of the new scheme, a laminar supersonic boundary layer on an adiabatic flat plate is calculated using first-order accuracy. The incoming Mach number is 2.0. The Reynolds number based on the length of the flat plate is 4×10^4. The Prandtl number of 1.0 is used in order to compare the numerical solutions with the analytical solution. The baseline mesh size is 41×31 in the direction along the plate and normal to the plate, respectively. The height of the computational domain is 0.82 times

Table 1 Computed nondimensional wall temperature using first-order schemes with the baseline mesh and refined meshes

Scheme	40 × 30	80 × 60	160 × 80	Error on fine mesh
Blasius	1.8000	1.8000	1.8000	0.0
Zha CUSP	1.8061	1.8022	1.8018	0.1%
Zha CUSP2	1.7980	1.7991	1.7988	−0.06%
Roe scheme	1.7990	1.8002	1.7996	−0.02%
Liou AUSM$^+$	1.7993	1.8000	1.8000	0.0
Van Leer	1.8157	1.8328	1.8333	1.8%
Van Leer–Hänel	1.7766	1.7970	1.7996	−0.02%

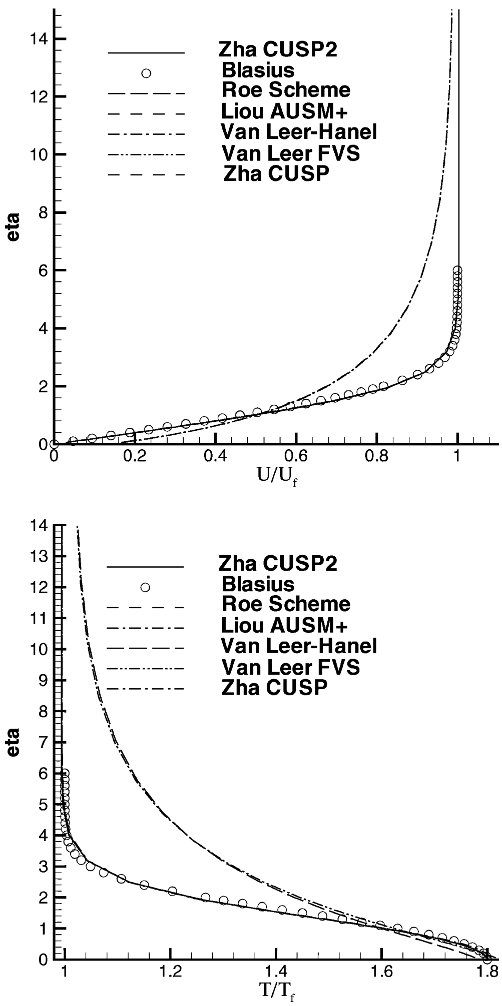


Fig. 2 Computed velocity and temperature profiles of a Mach 2.0 laminar boundary layer using first-order schemes.

of the flat-plate length. The mesh is stretched in the direction normal to the wall and there are 13 grid points within the boundary layer.

Figure 2 is the computed velocity and temperature profiles compared with the Blasius solution. The solutions of the Zha CUSP2 scheme, Roe scheme, AUSM$^+$ scheme,[9] and the original Zha CUSP scheme all agree accurately with the analytical solution. The Van Leer scheme[10] and the Van Leer–Hänel scheme[11] significantly thicken the boundary layer.

Table 1 shows the wall temperature predicted by all of the schemes using the baseline mesh and refined meshes. The Zha CUSP2 scheme, the Roe scheme, AUSM$^+$ scheme, and the original Zha CUSP scheme predict the wall temperature accurately with the very coarse baselines mesh of 40 × 30. For these four schemes, the baseline solutions are converged based on the mesh size. For the baseline coarse mesh, the original Zha CUSP scheme predicts the wall temperature with an error slightly larger than that of the modified scheme (Zha CUSP2). The largest error for the wall-temperature prediction using the baseline mesh is from the Van Leer–Hänel scheme. However, the Van Leer–Hänel scheme predicts the wall temperature precisely when the mesh is refined, and the solution is converged to the accurate solution. The Van Leer scheme converges to the wall temperature with an error as large as 1.8%, which is far greater than the results predicted by all of the other schemes with the error less than 0.1%. Even though the Van Leer–Hänel scheme predicts the wall temperature accurately on the refined mesh, the overall temperature and velocity profiles are as poor as that of the Van Leer scheme even on the refined mesh.[2] When the second-order schemes are used, both the velocity and temperature profiles of the Van Leer scheme and Van Leer–Hänel are improved (not shown).

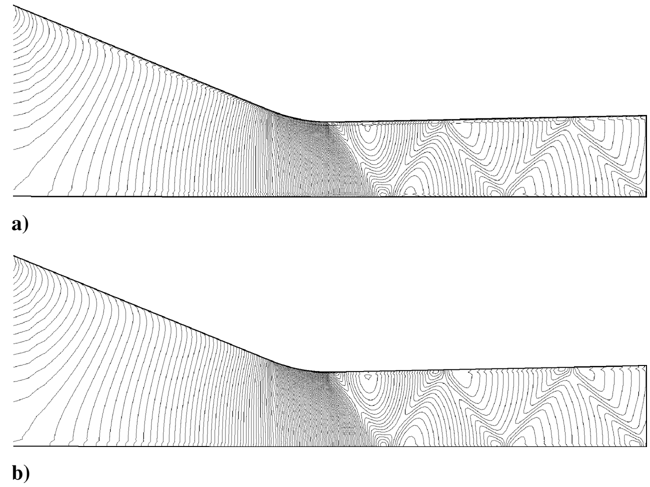


Fig. 3 Computed Mach-number contours using the a) Zha CUSP and b) Zha CUSP2 scheme.

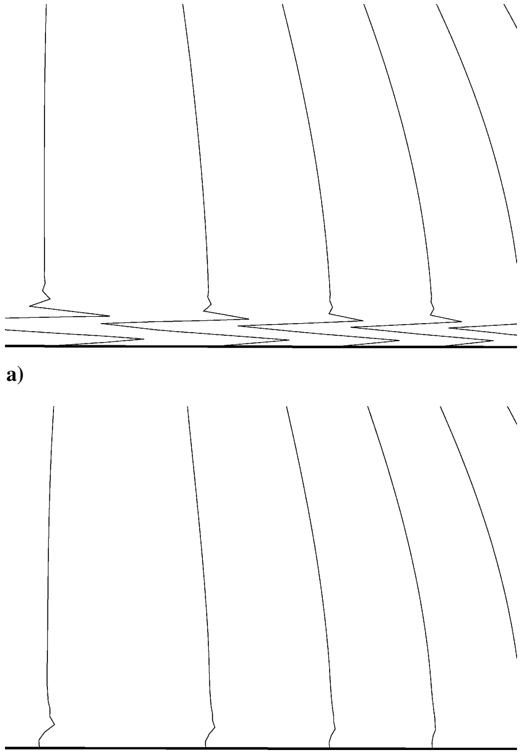
D. Transonic Converging-Diverging Nozzle

To examine the performance of the new scheme in two-dimensional flow and the capability to capture the shock waves that do not align with the mesh lines, a transonic converging-diverging nozzle is calculated as inviscid flow. The nozzle was designed and tested at NASA and was named as Nozzle A1.[12] Third-order accuracy of MUSCL-type differencing is used to evaluate the inviscid flux with no limiter.

Figure 3 is the computed Mach-number contours using the original Zha CUSP scheme and the Zha CUSP2 scheme with the mesh size of 175 × 80. The nozzle is symmetric about the centerline. Hence only the upper half of the nozzle is calculated. The upper boundary uses the slip wall boundary conditions, and the lower boundary of the centerline uses the symmetric boundary conditions. The Mach contour lines computed by the two schemes look very much the same. However, if the temperature contours near the wall are zoomed in, it can be seen that the temperature contours computed by the Zha CUSP scheme have large oscillations as shown in Fig. 4a. The temperature oscillations exist along the whole upper wall and lower symmetric boundary. The temperature oscillations are removed by the Zha CUSP2 scheme as shown in Fig. 4b. All of the other flow variables are smooth for the new scheme.

The reason that the original Zha CUSP scheme has no temperature oscillation shown in Fig. 2 for the flat plate, but has oscillation for the inviscid nozzle, might be the following: 1) the laminar Navier–Stokes equations provide some physical dissipation to smoothen the flow; 2) the boundary-layer gradient might generate some numerical dissipation to smoothen the flow; and 3) the first-order scheme (piecewise constant) is used for the flat plate. The first-order scheme is monotone and has higher numerical dissipation than the third-order scheme used for the nozzle.

The computed surface-pressure distribution of this nozzle agrees well with the experiment and is given in Ref. 4. The CPU time for the flux calculation is about $\frac{1}{4}$ of the time used by the Roe scheme, which is a significant CPU time saving. The scheme is also proved to be robust for three-dimensional calculations as indicated in Ref. 3.



a)

b)

Fig. 4 Zoomed temperature contours near the symmetric boundary using the a) Zha CUSP and b) Zha CUSP2 scheme.

IV. Conclusions

The original E-CUSP scheme suggested by Zha and Hu is modified to remove the temperature oscillation with low numerical dissipation. The pressure term in the energy equation dissipation of the original scheme is replaced by the total enthalpy in the modified scheme (Zha CUSP2).

For the one-dimensional Sod shock-tube problem, the modified scheme achieves crisper shock profile than that of the Roe scheme. The contact discontinuity is well resolved. For a quasi-one-dimensional transonic nozzle, the new scheme does not generate the expansion shock as the Roe scheme does.

For a Mach 2.0 supersonic adiabatic laminar flat-plate boundary layer, the new scheme is able to accurately resolve the boundary-layer velocity and temperature profiles using the first-order differencing. The solution is as accurate as that of the Roe scheme and the AUSM$^+$ scheme and hence demonstrates the low diffusion of the new scheme.

For a transonic converging-diverging nozzle, oblique shock waves and reflections are crisply captured even though the shock waves do not align with the mesh lines. The temperature oscillations generated by the original Zha CUSP scheme is removed.

Acknowledgments

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G. Candler
Associate Editor